International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2001.**

1. Find a polynomial $P(x)$ of degree 2001 such that $P(x) + P(1 - x) = 1$ for all real numbers x.

2. In a class with at least 5 students, each subject taken by a student results in pass or failure. For any arbitrarily chosen group of no less than 5 students, at least 80% of the failures received by this group are given to at most 20% of the students in the group. Prove that at least 75% of all the failures in this class are given to one student.

3. AD, BE and CF are the altitudes of triangle ABC. K, M and N are the respective orthocentres of triangles AEF, BFD and CDE. Prove that KMN and DEF are congruent triangles.

4. Each entry in two $m \times n$ tables A and B is either 0 or 1. The number of 1's in A is equal to the number of 1's in B. In both A and B, the numbers do not decrease from the left to the right in any row, nor from top to bottom in any column. For every k, $1 \leq k \leq m$, the sum of the entries in the top k rows of A is not less than the sum of entries in the top k rows of B. Prove that for any ℓ, $1 \leq \ell \leq n$, the sum of the numbers in the leftmost ℓ columns of B is at least the sum of the numbers in the leftmost ℓ columns of A.

5. In a chess tournament, every participant played with the others exactly once, getting 1 point for a win, $\frac{1}{2}$ for a draw and 0 for a loss. For each player, the sum of the points earned by the players who were beaten by this player was computed, as was the sum of the points earned by the players who beat this player.

   (a) Is it possible for the first sum to be greater than the second one for every player?

   (b) Is it possible for the first sum to be less than the second one for every player?

6. Prove that there exist 2001 convex polyhedra such that any three of them do not have any common points, and any two of them have at least one common boundary point but no common inner points.

7. Several boxes are placed along a circle. Each box may contain any number of chips, including zero. A move consists of taking all the chips from some box and placing them in the subsequent boxes clockwise, one chip in every box, beginning from the next box in the clockwise direction.

   (a) Suppose that in each move after the first one, one must take the chips from the box in which the last chip was placed on the previous move. Prove that after several moves, the initial distribution of the chips among the boxes will reappear.

   (b) Suppose that in each move, one may take the chips from any box. Is it true that for every initial distribution of the chips, one can get any possible distribution by performing an appropriate sequence of moves?

Note: The problems are worth 3, 5, 5, 5, 4+4, 8 and 4+4 points respectively.

Solution to Senior A-Level Spring 2001

1. Let $P(x) = x^{2001} - (1-x)^{2001} + \frac{1}{2}$. This is a polynomial of degree 2001 since the leading term is $2x^{2001}$. Now $P(x)+P(1-x) = x^{2001} - (1-x)^{2001} + \frac{1}{2} + (1-x)^{2001} + (1-(1-x))^{2001} + \frac{1}{2} = 1$ for all real numbers x.

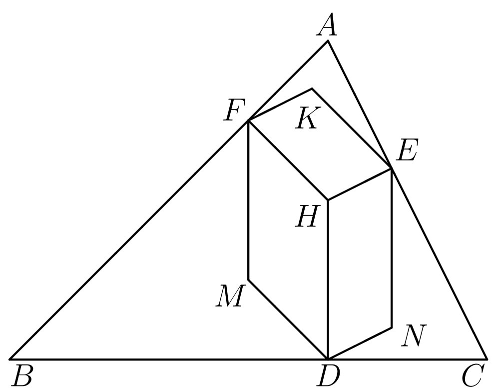
2. Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be the respective numbers of failures given to the n students in the class. For any k such that $k+4 \leq n$, we have $a_k \geq \frac{4}{5}(a_k + a_{k+1} + a_{k+2} + a_{k+3} + a_{k+4}$, or equivalently, $a_k \geq 3(a_{k+1} + a_{k+2} + a_{k+3} + a_{k+4})$. Suppose $n \equiv 1 \pmod 4$. Then

$$\begin{aligned} a_1 &\geq 4(a_2+a_3+a_4+a_5) \\ &\geq 3(a_2+a_3+a_4+a_5)+a_5 \\ &\geq 3(a_2+a_3+a_4+a_5)+4(a_6+a_7+a_8+a_9) \\ &\geq 3(a_2+a_3+\cdots+a_9)+a_9 \\ &\geq \cdots \\ &\geq 3(a_2+a_3+\cdots+a_n). \end{aligned}$$

This is equivalent to $a_1 \geq \frac{3}{4}(a_1+a_2+\cdots+a_n)$, which is the desired result. It does not matter if $n \not\equiv 1 \pmod 4$ as we simply take a_{n-4} as the extra term to generate the last inequality. For instance, if $n = 8$, we have

$$\begin{aligned} a_1 &\geq 3(a_2+a_3+a_4+a_5)+a_4 \\ &\geq 3(a_2+a_3+a_4+a_5)+4(a_5+a_6+a_7+a_8) \\ &\geq 3(a_2+a_3+\cdots+a_8). \end{aligned}$$

3. Let H be the orthocentre of triangle ABC. Note that FM and HD are both perpendicular to BC. Hence they are parallel to each other. Similarly, so are FH and MD. It follows that $DHFM$ is a parallelogram, Similarly, so is $DHEN$. Hence $FM = EN$ and they are parallel to each other. It follows that $EFMN$ is also a parallelogram, so that $EF = NM$. Similarly, we have $FD = KN$ and $DE = MK$. Hence triangles DEF and KMN are congruent.



4. Suppose the result is false. Let j be the smallest integer such that the sum of the numbers in the leftmost j columns of B is less than the sum of the numbers in the leftmost j columns of A. Then there are more 1's in column j of A than in column j of B. Let the number of 1's

in column j of A be i. Divide each of A and B into four quadrants, the dividing lines being between the $(m-i)$-th row and the $(m-i+1)$-st row and between the j-th column and the $(j+1)$-st column.

	j columns	$n-j$ columns
$m-i$ rows	II quadrant	I quadrant
i rows	III quadrant	IV quadrant

Let the numbers of 1's in the four quadrants of A be $a_1,\ a_2,\ a_3$ and a_4, and those of B be $b_1,\ b_2,\ b_3$ and b_4, respectively. Then $a_1+a_2+a_3+a_4=b_1+b_2+b_3+b_4$ and $a_2+a_3>b_2+b_3$, so that $a_1+a_4<b_1+b_4$. Since the last i numbers in column j of A are 1's, all of the numbers in the fourth quadrant of A are 1's. Hence $b_4 \le a_4$, so that $a_1<b_1$. Since the first $m-i$ numbers in column j of A and B are 0's, $a_2=b_2=0=$. Hence $a_1+a_2<b_1+b_2$, which is a contradiction.

5. Let there be n players. For player k, denote the first sum by w_k, the second sum by ℓ_k and the player's own score by s_k. Let $T=\sum_{k=1}^{n} s_k(w_k-\ell_k)$. We claim that $T=0$. Consider the game between players i and j. If it is a tie, it contributes nothing to T. Suppose i beats j. Then it contributes s_is_j to the term $s_i(w_i-\ell_i)$ while it contributes $-s_js_i$ to the term $s_j(w_j-\ell_j)$. The net contribution to T is still nothing, and the same holds by symmetry if j beats i. This justifies the claim.

   (a) Since $s_k \ge 0$ for all k, we cannot have $w_k>\ell_k$ for all k as otherwise $T>0$.

   (b) Since $s_k \ge 0$ for all k, we cannot have $w_k<\ell_k$ for all k as otherwise $T<0$.

6. We construct the polyhedra as follows. Start with an infinite inverted cone with vertex at the origin O and axis along the positive z-axis. On the surface of the cone are equally spaced lines $\ell_1,\ \ell_2,\ \ldots,\ \ell_{2001}$. For any positive number t, the plane $z=t$ intersects these lines at $A_1^t,\ A_2^t,\ \ldots,\ A_{2001}^t$. Let t_1 be an arbitrary positive number. Let D_1 be the disc which is the part of the plane $z=t_1$ inside the cone. Let B_1 be the midpoint of the minor arc $A_1^{t_1}A_2^{t_1}$. Let M_1 be the convex polygon $A_1^{t_1}A_2^{t_1}\ldots A_{2001}^{t_1}$. Consider the infinite oblique prism with base M_1 and lateral edges parallel to OB_1. This will eventually be truncated to yield a convex polyhedron P_1. For sufficiently large t, the distance between $A_1^tA_2^t$ and the midpoint of the minor arc $A_1^tA_2^t$ will be larger than the diameter of D_1 so that P_1 will not intersect the polygon $A_1^tA_2^t\ldots A_{2001}^t$. Let t_2 be such a value and let D_2 be the part of the plane $z=t_2$ inside the cone. Let B_2 be the midpoint of the minor arc $A_2^{t_2}A_3^{t_2}$. Let M_2 be the convex polygon $A_1^{t_2}A_2^{t-2}\ldots A_{2001}^{t_2}$ if it already touches P_1. If not, insert between $A_1^{t_2}A_2^{t_2}$ an additional vertex which lies on $D_2\cap P_1$. Let P_2 be the convex polyhedron obtained by eventually truncating

the oblique infinite prism with base M_2 and lateral edges parallel to OB_2. As before, for sufficiently large t, P_2 will not intersect the polygon $A_1^t A_2^t \dots A_{2001}^t$. Let T_3 be such a value, and we can continue the construction as before with M_3 being the convex polygon $A_1^{t_3} A_2^{t_3} \dots A_{2001}^{t_3}$, expanded if necessary, so that it touches both P_1 and P_2. For $1 \le i < j < k \le 2001$, P_i and P_j have only common boundary points on the plane $z = t_i$. Since P_k does not intersect this plane, P_i, P_j and P_k have no common points.

7. (a) We construct a directed graph as follows. Each vertex represents a distribution of chips coupled with the box from which chips must be taken. An arc goes from vertex A to vertex B if the state represented from A leads to the state represented by B. The out-degree of each vertex is clearly 1. We claim that so is its in-degree. From any given state, we can reverse the process by taking a chip from the box which is the last one to receive a chip, collecting 1 chip from each subsequent box counterclockwise, and stopping when we reach an empty box. We can only deposit all the chips into this box, and mark it as the one from which chips must be taken. This is the unique state which leads to the given one, and the claim is justified. Since the number of vertices is finite, the directed graph is a union of disjoint directed cycles. It does not matter which cycle contains the vertex which represents the initial state, as all states represented by vertices in this cycle will reappear.

   (b) We construct a directed graph as follows. Each vertex represents a distribution of the chips. An arc goes from vertex A to vertex B if it is possible to change the distribution represented by A directly into the distribution represented by B. The out-degree and the in-degree of each vertex are both equal to the number of non-empty boxes in the distribution represented by that vertex. Hence the directed graph consists of strongly connected components. It has only one component because the distribution in which all chips are in a specific box is reachable from any other distribution. We simply do not take chips from that box. Hence we can change any distribution into any other distribution.