International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2001.**

1. At 12:20 pm, a bus leaves on a 100-kilometre route. It is equiped with a computer, which makes the following announcement at 1:00 pm, 2:00 pm, 3:00 pm, 4:00 pm, 5:00 pm and 6:00 pm: "Assuming that the average speed for the remaining part of the route is the same as that in the part already covered, the bus will arrive at its destination in another hour." Can it be right, and if so, how many kilometers has the bus covered at 6 pm?

2. The cube of an n-digit number has m digits. Is $n + m = 2001$ possible?

3. In triangle ABC, X is a point on AB and Y is a point on BC. The segments AY and CX intersect at Z. If $AY = YC$ and $AB = ZC$, prove that the points B, X, Z and Y lie on a circle.

4. On a 3×100 board, two players alternately place dominoes. The first player places them as 1×2 rectangles while the second places them as 2×1 rectangles. The loser is the one who cannot make a move. Which player can force a win, and what is the winning strategy?

5. Nine points are drawn on the surface of a regular tetrahedron of edge 1 centimetre. Prove that among these points there are two such that the distance in space between them is at most 0.5 centimetre.

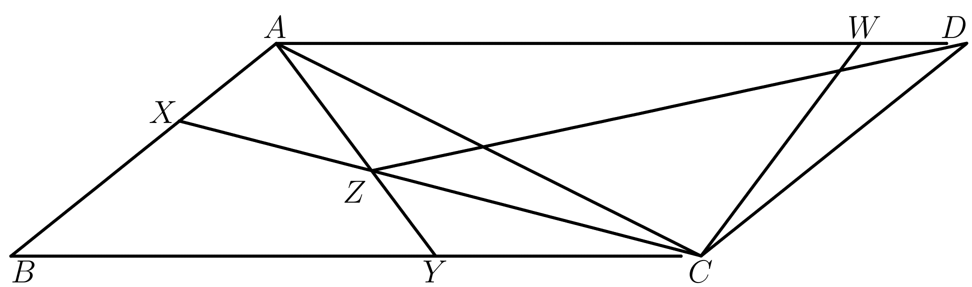
Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

Solution to Senior O-Level Spring 2001

1. The computer can be right if at $k:00$ pm, the distance covered is $\frac{60k-20}{60k+40}$ of the total distance for $1 \leq k \leq 6$. At $k:00$ pm, the average speed so far is $\frac{1}{60k+40}$, and $\frac{60}{60k+40}$ of the distance is still to be covered, requiring another hour at the same speed. Note that between $(k-1):00$ pm and $k:00$ pm, the distance coverd is $\frac{60k-20}{60k+40} - \frac{60k-80}{60k-20} = \frac{80}{(60k+40)(60k-20)}$ of the total distance, and these quantites are all positive. At 6:00 pm, $\frac{340}{400} = \frac{17}{20}$ of the distance has been covered, which translates into 85 kilometres.

2. Note that 10^{500} has 501 digits while $(10^{500})^3$ has 1501 digits for a total of 2002. For any integer $a > 10^{500}$, a and a^3 will also have at least 2002 digits between them. If $a < 10^{500}$, then it has at most 500 digits while $a^3 < 10^{1500}$ has at most 1500 digits. Thus it is not possible for the sum of the numbers of digits to be 2001.

3. First, note that $CY = AY < CZ = AB$. This is evident if AY is perpendicular to BC. If not, one of $\angle AYB$ and $\angle AYC$ will be obtuse. In the former case, $AB > AY$. In the latter case, $CZ > CY$.



Complete the parallelogram $ABCD$ and let W on AD be such that $\angle WCY = \angle AYC$. Then $CY = CW$, $CZ = AB = CD$ and $\angle AYC = \angle WCY = \angle DWC$. Since $AB > AY$, triangles ZYC and DWC are congruent, so that $\angle AYC = \angle WCY = \angle DCZ$. It follows that triangles YAC and DZC are similar, so that $\angle ZAC = \angle ZDC$. Hence $ADCZ$ is a cyclic quadrilateral. Now $\angle XBY + \angle XZY = \angle ADC + \angle AZC = 180°$, so that $BXZY$ is also cyclic.

4. We first play the game on a 3×4 board. The first player can force a win by placing the first domino anywhere on the middle row. This leaves the second player with two columns on which dominoes can be placed, while the first player can place two more dominoes above and below the first one. Now divide the 3×1000 board into 250 3×4 boards. Since the second player can only place dominoes within such a board, the first player can force a win by adopting the earlier strategy on each 3×4 board.

5. Connect the midpoints of two edges of the tetrahedron if and only if they share a common vertex. This divides the solid tetrahedron into four corner tetrahedra plus a central octahedron. If two of the nine points lie on the surface of one of those corner tetrahedra, then the distance between them is at most 0.5 centimetre. Hence we may assume that each contains at most one point, leaving five on the surface of the central octahedron. Only four of its faces are on the surface of the original tetrahedron. By the Pigeonhole Principle, two of the five points will lie on the same face, and the distance between them is at most 0.5 centimetre.