International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2001.**

1. In a certain company, 10% of the employees get 90% of the total salary. The company consists of several branches. Is it possible that in each branch, the total salary of any 10% of the employees is at most 11% of the total salary paid in this branch?

2. There are three piles of stones, with 51, 49 and 5 stones respectively. In a move, any two piles may be combined into one pile. Alternatively, any pile with an even number of stones may be divided into two equal piles. Is it possible to obtain 105 piles with 1 stone in each after a sequence of such moves?

3. The point A lies inside $\angle KMN$. The points B and C lie respectively on KM and MN. If $\angle CBM = \angle ABK$ and $\angle BCM = \angle ACN$, prove that the circumcentre of triangle BCM lies on the line AM.

4. A convex polygon is divided into triangles by diagonals which do not intersect except at the vertices of the polygon. Each vertex is labelled with the number of triangles to which it belongs. Is it possible to reconstruct all the diagonals using these numbers if the diagonals are erased?

5. A black pawn and a white pawn are placed on a chessboard. In each move, one of the pawns goes to an adjacent vacant square of the chessboard either vertically or horizontally. It is desired to construct a sequence of moves so that every possible position of the two pawns on the chessboard will appear exactly once.

   (a) Is this possible if the pawns must be moved alternately?

   (b) Is this possible if the pawns need not be moved alternately?

6. AD, BE and CF are the altitudes of triangle ABC. K, M and N are the respective orthocentres of triangles AEF, BFD and CDE. Prove that KMN and DEF are congruent triangles.

7. Alex picks an integer greater than 9 and less than 100. Gregory is trying to guess this integer by naming its two digits. If the integer named by Gregory is correct, or if one digit is correct and the other differs from its correct value by one, Alex will say “hot”; otherwise he will say “cold”. For example, if the number 65 is the one picked by Alex, then the answer is “hot” if and only if Gregory names 65, 64, 66, 55 or 75.

   (a) Prove that Gregory has no strategy guaranteeing that he will deduce Alex’s integer in at most 18 attempts.

   (b) Find a strategy for Gregory to deduce Alex’s integer, regardless of what it is, using at most 24 attempts.

   (c) Is there a strategy for Gregory to do so in at most 22 attempts?

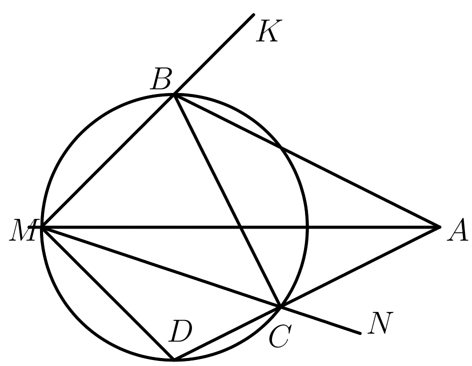
Note: The problems are worth 3, 5, 5, 5, 3+4, 7 and 2+3+3 points respectively.

Solution to Junior A-Level Spring 2001

1. Let the company have two branches. The first branch consists of 10% of all the employees but receiving 90% of the total salary. The second branch consists of the remaining 90% of the employees receiving the remaining 10% of the total salary. Within each branch, all employees have equal pay. Then 90% of the total salary does go to 10% of the employees, and yet in each branch, any 10% of the employees get exactly 10% of the totaly salary paid in that branch.

2. Since all three piles initially contain odd numbrs of stones, the first move must be a merge. If we merge the piles with 5 and 49 stones, then we have two piles in each of which the number of stones is a multiple of 3. We claim that from now on, the number of stones in any pile must be a multiple of 3. Since 1 is not a multiple of 3, not even 1 pile with 1 stone can be obtained. Note that the merger of two piles of stones each with a number of stones equal to a multiple of 3 will result in one pile with a number of stones equal to a multiple of 3. Splitting of a pile with an even number of stones does not change this property. Thus the claim is justified. Similarly, if we begin by merging the piles with 5 and 51 stones, then the number of stones in every subsequent pile must be a multiple of 7. If we begin by merging the piles with 49 and 51 stones, then the number of stones in every subsequent pile must be a multiple of 5. It follows that we cannot get 105 piles each with 1 stone.

3. The result is trivial if $MB = MC$. Hence we assume that $MB < MC$. Let D be the point on the circumcircle of triangle MBC such that $MB = MD$. Then $\angle ACN = \angle BCM = \angle DCM$, so that A, C and D are collinear. Now $\angle MDC = \angle CBK = \angle CBA + \angle ABK = \angle CBA + \angle MBC = \angle MBA$. Moreover, this angle is obtuse. Since $MD = MB$ and $AM = AM$, triangles MAD and MAB are congruent. By symmetry, the circumcentre of triangle MBC lies on AM.



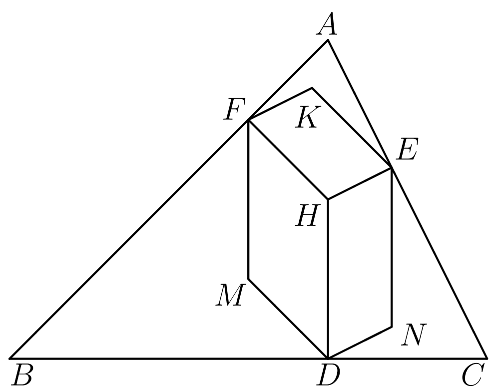
4. We use induction on the number n of sides of the convex polygon. For $n = 3$, we have a trivial reconstruction. Suppose reconstruction is possible for some $n \geq 3$. Consider an $(n+1)$-gon with erased diagonals. At least one vertex must belong to a single triangle and is labelled 1. Draw the diagonal connecting its two neighbours. The remaining part is then a convex n-gon, with labels suitably adjusted, and reconstruction can be continued by the induction hypothesis.

5. (a) This is impossible. For any position of the black pawn, there are 63 possible positions for the white pawn. Since the pawns move alternately, each visit of the black pawn to this square accounts for 2 of the 63 possible positions for the white pawn. Since 63 is

odd, this means that the black pawn must either end its moves there or begin there. It is clearly not possible for the black pawn to do so in every square.

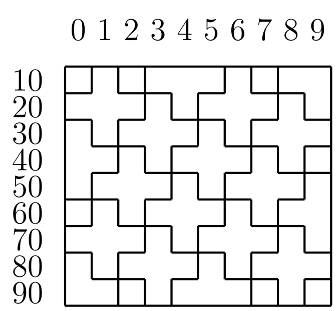
(b) This is still impossible. Define a position as even if both pawns are on squares of the same colour, and odd otherwise. Even if alternate moves are not required, the positions must necessarily be alternately odd and even. The number of even positions is $2\binom{32}{2} = 992$ while the number of odd positions is $32^2 = 1024$. If all even positions appear, then some odd positions must appear more than once.

6. Let H be the orthocentre of triangle ABC. Note that FM and HD are both perpendicular to BC. Hence they are parallel to each other. Similarly, so are FH and MD. It follows that $DHFM$ is a parallelogram, Similarly, so is $DHEN$. Hence $FM = EN$ and they are parallel to each other. It follows that $EFMN$ is also a parallelogram, so that $EF = NM$. Similarly, we have $FD = KN$ and $DE = MK$. Hence triangles DEF and KMN are congruent.



7. (a) Since each guess covers at most 5 numbers, it takes at least 18 guesses to cover all 90 numbers, even if this can be done with no overlap. However, if all answers other than the last one are "cold", there are no more guess to nail down the exact number.

(b) We represent the 90 numbers by a 9×10 grid, which is partitioned into 18 crosses, some incomplete, along with 8 isolated squares. These squares represent the numbers 10, 12, 17, 39, 50, 92, 97 and 99. The centres of the crosses represent the first 18 guesses. If any of the answers is "hot", we certainly have enough guesses left to nail down the exact number. Suppose all answers are "cold". The next two guesses will be 11 and 98. If either answer is "hot", we can nail down the exact number. If not, the next three guesses will be 18, 39 and 70. We will know Alex's number for sure in 23 moves.



(c) By making minor adjustments to the crosses near the top right and bottom left corners, we can save 1 move. Again, assume that the first 18 quesses are answered with "cold".

The next three guesses will be 11, 39 and 70. If any of the answers is "hot", we have a more guess that will make it work. If all the answers are "cold", the guess 96 will tell us whether Alex's number is 97 or 99.