International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2001.**

1. We may replace the positive integer n by ab where a and b are positive integers such that $a + b = n$. Can 2001 be obtained from 22 by a sequence of such replacements?

2. Let D, E and F be the midpoints of BC, CA and AB respectively. If one of DE, EF and FD is longer than one of AD, BE and CF, prove that ABC is an obtuse triangle.

3. Twenty kilograms of cheese are on sale in a food store, and customers are queued up to buy some of it. After each of the first ten customers has made her purchase, the sales girl announces, "If everyone buys an amount equal to the average amount bought by customers already served, then there is just enough cheese left for ten more customers." Can she be right, and if so, how much cheese will be left after the first ten customers have made their purchases?

4. There are five identical paper triangles on a table. Each may be slided in any direction parallel to itself without rotation.
   (a) Is it true that any one of them can be covered by the other four?
   (b) Prove that any one of them can be covered by the other four if the triangle is equilateral.

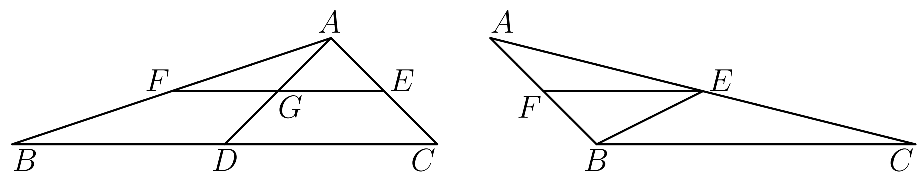
5. On a 15×15 board are 15 rooks that do not attack one another. Then each rook makes one move like that of a knight. Prove that after this is done, a pair of rooks will necessarily attack each other.

Note: The problems are worth 3, 4, 4, 2+3 and 5 points respectively.

Solution to Junior O-Level Spring 2001

1. Working backwards, we see that $2001 = 3 \times 667$ can be obtained from 3+667=670, that $670 = 10 \times 67$ can be obtained from 10+67=77 and that $77 = 7 \times 11$ can be obtained from 7+11=18. From any $n = 1 + (n-1)$, we can obtain $n - 1 = 1(n-1)$. Hence starting from 22, we can obtain in succession 21, 20, 19, 18, 77, 670 and 2001.

2. Suppose first that EF is longer than AD. Let G be the point of intersection of AD and EF. Then $AG < EG = FG$. Hence $\angle GAE > \angle GEA$ and $\angle GAF > \angle GFA$. Since the sum of these four angles is 180°, $\angle CAB = \angle GAE + \angle GAF > 90°$. Suppose now that EF is longer than BE. Then $\angle ABC > \angle EBF > \angle EFB = \angle FAE + \angle FEA = \angle CAB + \angle BCA$. Hence $\angle ABC > 90°$. All other cases are equivalent to either of these two by symmetry.



3. The sales girl can be right if the first k customers buy $\frac{k}{10+k}$ of the cheese for $1 \le k \le 10$. After the k-th purchase, the average amount sold to each customer so far is $\frac{1}{10+k}$ of the cheese, and $\frac{10}{10+k}$ of it remains. Thus there is just enough for ten more customers. Note that the k-th customer buys $\frac{k}{10+k} - \frac{k-1}{9+k} = \frac{10}{(10+k)(9+k)}$ of the cheese, and these quantites are all positive. After the 10-th purchase, $\frac{10}{10+10} = \frac{1}{2}$ of the cheese has been sold, meaning that ten kilograms are left.

4. (a) Let ABC be a triangle where B is very close to the midpoint of CA. Rotate this triangle about C through angles of 72°, 144°, 216° and 288°, obtaining four other triangles congruent to it. Since each can cover a very small portion of the long side of any of the other triangles, none can be covered by the other four by translation only.

   (b) Divide the chosen equilateral triangle into four small equilateral triangles by segments joining the midpoints of its sides. The circumcircle of the small triangle in the middle coincides with the incircle of the large triangle. We now slide each of the other four large triangles so that the incircle of each coincides with the circumcircle of a different one of the four small triangles. Then the chosen triangle is completely covered by the other four.

5. Record the row numbers and column numbers of each rook. Since no two attack each other, the row numbers are all distinct, as are the column numbers. Hence among these 30 numbers, 16 are odd and 14 are even. When a rook makes a knight move is made, either its row number changes by 1 and its column number changes by 2, and vice versa. In all, 15 of the 30 numbers preserve their parity while the remaining 15 change theirs. Thus it is not possible to have 16 odd and 14 even numbers among them after the moves. This means that two of the rooks must attack each other.